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Self-similarity of quasilattices in two dimensions: III. Inflation by a non-unit PV number

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Abstract. It is shown that every n-gonal quasilattice in two dimensions has self-similarity with a complex self-similarity ratio which is equal to a non-unit PV number in the n-cyclotomic field. In this self-similarity, the original quasilattice can be inflated to a number of different quasilattices which are locally isomorphous to each other. The number is equal to the algebraic norm of the relevant PV number in the n-cyclotomic field. Several examples are presented for the cases of octagonal, decagonal and dodecagonal quasilattices. The difference between the present self-similarity and that associated with a PV unit is clarified.

1. Introduction

We have shown in previous papers (Niizeki 1989a, b, hereafter referred to as I and II, respectively) than, for even n (≥ 8), an n-gonal quasilattice in two dimensions (2D) has self-similarity with a complex self-similarity ratio being equal to a PV unit in the n-cyclotomic field $\boldsymbol{Q}(\zeta)$, $\zeta = \zeta_n \equiv \exp(2\pi i/n)$; a PV unit τ is an algebraic integer satisfying (i) τ^{-1} is also an algebraic integer, (ii) $|\tau| > 1$ and (iii) $|\tau'| < 1$ with τ' being any conjugate but $\bar{\tau}$ (the complex conjugate) of τ in $\boldsymbol{Q}(\zeta)$. The lattice points of the n-gonal quasilattice are represented by points (complex numbers) in the complex plane $\boldsymbol{C}$ and τ acts multiplicatively onto $\boldsymbol{C}$.

All the PV units in $\boldsymbol{Q}(\zeta_n)$ form a commutable semigroup which is generated by a finite number of the fundamental PV units. The number of the fundamental PV units is one if $n = 8$, 10, or 12; the respective fundamental PV unit is $1+\sqrt{2}$, $(1+\sqrt{5})/2$ or $1+\zeta_{12}$. The number of the fundamental units is two or more if $n \geq 14$.

On the other hand, it is reported that an octagonal quasilattice has an additional self-similarity with ratio $2+\sqrt{2}$ (Watanabe *et al* 1987), which is a PV number in $\boldsymbol{Q}(\zeta_8)$ but not a unit; a PV number is an algebraic integer satisfying the conditions (ii) and (iii) above. An n-gonal quasiperiodic pattern constructed with the Pleasants method has also self-similarity with a ratio being equal to a non-unit PV number (Pleasants 1984). In this paper, we will show that every n-gonal quasilattice is, in fact, self-similar with respect to inflations with ratios given not only by a PV unit but also a non-unit PV number. The semigroup formed by all the PV numbers in $\boldsymbol{Q}(\zeta_n)$ has an infinite number of generators, so that an n-gonal quasilattice has an infinite variety of self-similarities.

A quasilattice is a structural model of a quasicrystal. Self-similarity of the quasilattice may have a profound effect on the physical properties of the relevant quasicrystal.

Therefore, it is important to investigate the difference (if any) in self-similarity between the case of a PV unit as investigated in I and II and the case of a non-unit PV number to be investigated in this paper.

In § 2, we introduce an n-gonal lattice in a higher dimension and present a systematic method of dividing the n-gonal lattice into equivalent sublattices. Most of this section will be a resumé of some results presented in I and II. In § 3, we show self-similarity of an n-gonal quasilattice with respect to a non-unit PV number. In § 4, we apply the theory in § 3 to n-gonal quasilattices with $n \leqslant 12$. In § 5, we contrast the feature of self-similarity presented in this paper with that presented in I and II. We discuss, also, a related subject.

2. An n-gonal lattice and its division into several equivalent sublattices

Let n ($n \geqslant 8$) be an even integer and let r be the rotation of the plane (the two-dimensional Euclidean space) E_2 by $2\pi/n$ with respect to the origin. Then, the cyclic group C_n generated by r is a point group with order n. E_2 is identified with $\boldsymbol{C}$, the complex plane, $E_2 \simeq \boldsymbol{C}$, and, then, r is equivalent to a multiplication of a complex number $\zeta = \zeta_n$ onto $\boldsymbol{C}$; $C_n \simeq \hat{C}_n \equiv \{1, \zeta, \ldots, \zeta^{n-1}\}$. ζ is an algebraic integer which satisfies the equation $P_n(x) = 0$ with $P_n(x)$ being the n-cyclotomic polynomial. The order of $P_n(x)$ is given by $\phi(n)$ with ϕ being the Eulerian function in number theory. $\phi(n)$ is an even integer and we put $m \equiv \phi(n)/2$. $1, \zeta, \ldots, \zeta^{2m-1}$ are linearly independent over $\boldsymbol{Z}$. ζ has $2m$ conjugates including itself, $\zeta, \zeta', \ldots, \zeta^{(2m-1)}$. We can assume that $\zeta^{(m+k)}$ is the complex conjugate of $\zeta^{(k)}$ for $k = 0, 1, \ldots, m-1$.

Let $\boldsymbol{a}_k = (\zeta^k, (\zeta')^k, \ldots, (\zeta^{(m-1)})^k)$, $k = 0, 1, \ldots, 2m-1$, be m-dimensional complex vectors. Then, they are linearly independent over the real field. They form a set of basis vectors of a $2m$-dimensional Euclidean space $E_{2m} \simeq \boldsymbol{C}^m = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}$. A $2m$-dimensional lattice L generated by the $2m$ basis vectors, $L = \{n_0\boldsymbol{a}_0 + n_1\boldsymbol{a}_1 + \ldots + n_{2m-1}\boldsymbol{a}_{2m-1} \mid n_k \in \boldsymbol{Z}\}$, is an n-gonal lattice; L is invariant against a point symmetry group $\tilde{D}_n$ which is isomorphous to the dihedral group D_n. Each component in $\boldsymbol{C}^m$ in an invariant subspace against $\tilde{D}_n$.

Let π be the projector which projects $E_{2m} \simeq \boldsymbol{C}^m$ onto the first subspace $\boldsymbol{C}$ in $\boldsymbol{C}^m$. Then, $\pi(\boldsymbol{a}_k) = \zeta^k$, $k = 0, 1, \ldots, 2m-1$, and, accordingly, π is a bijection (a one-to-one mapping) from L onto $\boldsymbol{Z}(\zeta) \equiv \{n_0 + n_1\zeta + \ldots + n_{2m-1}\zeta^{2m-1} \mid n_k \in \boldsymbol{Z}\}$, which is the ring of all the algebraic integers in $\boldsymbol{Q}(\zeta)$. $\pi(\tilde{D}_n) = \hat{C}_n + \sigma\hat{C}_n \simeq D_n$, where σ is the complex conjugate operation, i.e. the reflection with respect to the real axis of $\boldsymbol{C}$.

In this paper we assume for simplicity that the class number of $\boldsymbol{Q}(\zeta)$ is one, so that every ideal of $\boldsymbol{Z}(\zeta)$ is generated by one generator. This assumption is satisfied if $n \leqslant 44$, so that all the important cases are included.

Let $\boldsymbol{J}$ be an ideal of $\boldsymbol{Z}(\zeta)$. Then $\boldsymbol{J}$ is called self-conjugate if $\bar{\boldsymbol{J}}$ (the complex conjugate of $\boldsymbol{J}$) is identical to $\boldsymbol{J}$. We shall refer to a generator of a self-conjugate ideal as a D_n-conjugate number. A necessary and sufficient condition for an algebraic integer μ to be D_n-conjugate is that $\mu = \bar{\mu}\zeta^k$ for some integer k.

Let J be a self-conjugate ideal of $\boldsymbol{Z}(\zeta)$. Then $K \equiv \pi^{-1}(\boldsymbol{J})$ is a superlattice of L and, moreover, is invariant against $\tilde{D}_n$. K is called by this property as a D_n-superlattice of L. The number of the lattice points included in a unit cell of K is given by $q = N\boldsymbol{J}$, the norm of the ideal $\boldsymbol{J}$. Let μ be a generator of $\boldsymbol{J}$, i.e. $\boldsymbol{J} = \mu\boldsymbol{Z}(\zeta)$. Then, $N\boldsymbol{J} = N(\mu) \equiv \mu\mu' \ldots \mu^{(2m-1)}$, where $\mu', \mu'', \ldots, \mu^{(2m-1)}$ are the conjugates of μ in $\boldsymbol{Q}(\zeta)$. Note that L is transformed to K by a linear transformation $\tilde{\mu}$ which is represented by a diagonal complex matrix $(\mu, \mu', \ldots, \mu^{(m-1)})^{\text{diag}}$ acting on $C^m \simeq E_{2m}$. Therefore, we may write $\tilde{\mu} = \pi^{-1}(\mu\cdot)$, where $\mu\cdot$ is a linear transformation defined by a multiplication of μ onto C.

A necessary and sufficient condition for μ to be a unit in $Q(\zeta)$ is given by $\mu Z(\zeta) = Z(\zeta)$ or, equivalently, $N(\mu) = 1$. If μ is not a unit, $N(\mu) > 1$.

Two algebraic integers μ and ν are called associates of each other if they generate an identical ideal, i.e. $\mu Z(\zeta) = \nu Z(\zeta)$. A necessary and sufficient condition for μ and ν to be associates of each other is that $\mu = \varepsilon\nu$ with ε being a unit in $Q(\zeta)$.

Let μ be a generator of a self-conjugate ideal J of $Z(\zeta)$ and let ε be a PV unit in $Q(\zeta)$. Then $\varepsilon^k\mu$ is a PV number for a sufficiently large integer k. Thus, every ideal has a PV generator, i.e. a generator which is a PV number. Let τ be one of the PV generators of J with the smallest magnitude. Then, any other PV generator of J is written as $\varepsilon\tau$ with ε being a unit such that $|\varepsilon| \geq 1$. Note that $|\varepsilon\tau| = |\tau|$ only when $\varepsilon = 1$, $\zeta, \ldots$ or ζ^{n-1}. We shall call τ (and also $\zeta^k\tau$ for any k) as a fundamental PV generator of J. One of the fundamental PV generators of a self-conjugate generator satisfies $\tau = \bar{\tau}$ (real) or $\tau = \bar{\tau}\zeta$.

Let J be a self-conjugate ideal of $Z(\zeta)$. Then, L is divided into q $(= NJ)$ sublattices which are identical to $K = \pi^{-1}(J)$ except translations. Each sublattice is labelled by an element in $\Lambda = Z(\zeta)/J$, the residue class ring;

$$L = \bigcup_{\lambda \in \Lambda} K(\lambda) \tag{1}$$

where $K(\lambda)$ denotes the sublattice labelled by λ $(K(0) = K)$. By definition, Λ can be embedded in $Z(\zeta)$, i.e. $\Lambda \subset Z(\zeta)$. Then, $K(\lambda)$ is given with $l(\lambda) = \pi^{-1}(\lambda) \in L$ as $l(\lambda) + K = \{l(\lambda) + x \mid x \in K\}$. The q sublattices are transformed (permuted) among themselves by a symmetry operation in $\tilde{\mathrm{D}}_n$.

3. Self-similarity of an *n*-gonal quasilattice

3.1. The case of a 'Bravais-type' quasilattice

We divide C^m into the internal and the external spaces as $C^m = C \oplus C^{m-1}$, respectively, where C, the external space, is the first component in C^m, i.e. $C = \pi(C^m)$, and C^{m-1} is the orthogonal complement of C in C^m. Both the spaces are invariant subspaces of C^m against $\tilde{\mathrm{D}}_n$. C is irreducible but C^{m-1} is reducible unless $m = 2$. We shall denote the projection of C^m onto C^{m-1} by π'. The restriction of $\tilde{\mathrm{D}}_n$ onto C^{m-1} is a point group D'_n $(= \pi'(\tilde{\mathrm{D}}_n))$ which is isomorphous to D_n.

We introduce here a window W which is a convex polygon (or polytope if $m > 2$) or a star-like polygon (or polytope) in the internal space C^{m-1} and is invariant against D'_n. Then, we can construct with the projection method a 'Bravais-type' n-gonal quasilattice whose macroscopic point symmetry is equal to D_n;

$$L_Q(\boldsymbol{\phi}, W) = \{\pi(z) \mid z \in L \text{ and } \pi'(z) \in \boldsymbol{\phi} + W\} \tag{2}$$

where $\boldsymbol{\phi}$ is an arbitrary vector, a so-called phase vector, in C^{m-1}. Two quasilattices with a common window but with different phase vectors belong to the same local-isomorphism class (LI class).

Let J be a self-conjugate ideal of $Z(\zeta)$ and assume that L is divided as (1) in terms of a D_n-superlattice $K = \pi^{-1}(J)$. Then, $L(\boldsymbol{\phi}, W)$ in (2) is divided into equivalent sublattices as

$$L_Q(\boldsymbol{\phi}, W) = \bigcup_{\lambda \in \Lambda} L_Q^{(\lambda)}(\boldsymbol{\phi}, W) \tag{3a}$$

$$L_Q^{(\lambda)}(\boldsymbol{\phi}, W) = \{\pi(z) \mid z \in K(\lambda) \text{ and } \pi'(z) \in \boldsymbol{\phi} + W\}. \tag{3b}$$

In order to avoid a complication, we shall confine our argument hereafter to the case of $m = 2$; an extension of the result to the case $m \geqslant 2$ is straightforward. Then, the internal space as well as the external space is two-dimensional and we shall denote the former by $\boldsymbol{C}'$ so that it is distinguished from the latter.

The condition $z \in K(\lambda)$ $(= \boldsymbol{l}(\lambda) + K)$ is equivalent to $\pi(z) \in \lambda + \boldsymbol{J}$. Therefore, (3$b$) can be rewritten as

$$L_Q^{(\lambda)}(\phi, W) = \{\nu \mid \nu \in \lambda + \boldsymbol{J} \text{ and } \nu' \in \phi + W\} \tag{3c}$$

where ν' denotes a conjugate but $\bar{\nu}$ of ν in $\boldsymbol{Q}(\zeta)$.

Let τ be a PV generator of $\boldsymbol{J}$, $\boldsymbol{J} = \tau \boldsymbol{Z}(\zeta)$. Then, τ acts on $\boldsymbol{C}$ as an expansive linear transformation and its conjugate τ' on $\boldsymbol{C}'$ as a contractive one. It follows that $\tau' W \subset W$ provided that W is sufficiently close to a disc in $\boldsymbol{C}'$. Even if it does not hold, we can take another PV generator of $\boldsymbol{J}$ so that it does hold. Then, we obtain $L_Q^{(\lambda)}(\phi, W) \supset L_Q^{(\lambda)}(\phi, \tau' W)$. On the other hand, using the equivalence

$$\nu \in \lambda + \boldsymbol{J} \leftrightarrow \nu = \lambda + \tau\kappa \text{ and } \kappa \in \boldsymbol{Z}(\zeta)$$

we obtain that

$$\begin{aligned} L_Q^{(\lambda)}(\phi, \tau' W) &= \{\lambda + \tau\nu \mid \nu \in \boldsymbol{Z}(\zeta) \text{ and } \lambda' + \tau'\nu' \in \phi + \tau' W\} \\ &= \lambda + \tau L_Q(\phi_\lambda, W) \qquad \text{with } \phi_\lambda = (\phi - \lambda')/\tau'. \end{aligned}$$

It follows that $L_Q^{(\lambda)}(\phi, W)$ has a sublattice (subset) being similar to $L_Q(\phi_\lambda, W)$, which belongs to the same LI class as the original quasilattice to which $L_Q(\phi, W)$ belongs. Consequently, we can conclude that $L_Q(\phi, W)$ has self-similarity associated with a non-unit PV number. However, the quasilattice is not uniquely inflated in this case but can be inflated into q different but equivalent quasilattices, which is markedly different from the case of self-similarity associated with a PV unit as discussed in I. This is because $\tilde{\tau} = \pi^{-1}(\tau \cdot)$ leaves L invariant if τ is a PV unit but changes L to a superlattice if τ is a non-unit PV number.

We shall summarise the inflation rule of the self-similarity discussed above: (i) take a self-conjugate ideal $\boldsymbol{J} \subset \boldsymbol{Z}(\zeta)$ and divide the n-gonal quasilattice into equivalent sublattices labelled by the elements in $\Lambda = \boldsymbol{Z}(\zeta)/\boldsymbol{J}$; (ii) only one of the q $(= N\boldsymbol{J})$ sublattices is retained; and (iii) take a PV generator τ of $\boldsymbol{J}$ satisfying $\tau' W \subset W$ and narrow the window W used in selecting acceptable lattice points to $\tau' W$.

If τ is not real, τ satisfies $\tau = \bar{\tau}\zeta$ and $\arg \tau = \pi/n$, so that the directions of the bonds in the inflated quasilattices are rotated by π/n from those in the original quasilattice.

3.2. The case of a 'non-Bravais-type' quasilattice

We shall consider the case of a 'non-Bravais-type' n-gonal quasilattice, which is given, as shown in II, by

$$L_Q(\phi, \{W\}) = \bigcup_{\lambda \in \Lambda} L_Q^{(\lambda)}(\phi, W(\lambda)) \tag{4a}$$

$$L_Q^{(\lambda)}(\phi, W(\lambda)) = \{\nu \mid \nu \in \lambda + \boldsymbol{J} \text{ and } \nu' \in \phi + W(\lambda)\} \tag{4b}$$

where $\{W\}=\{W(\lambda)|\lambda\in\Lambda\}$ is the set of windows assigned to the sublattices $K(\lambda)$. We assume that $W(0)$ is the largest window among the $W(\lambda)$. Obviously, $L_Q(\phi,\{W\})\supset L_Q^{(0)}(\phi, W(0))$. Let τ be a PV generator of $\boldsymbol{J}$. Then

$$\begin{aligned} L_Q^{(0)}(\phi, W(0)) &= \{\nu|\nu\in\tau\boldsymbol{Z}(\zeta) \text{ and } \nu'\in\phi+W(0)\} \\ &= \tau\{\nu|\nu\in\boldsymbol{Z}(\zeta) \text{ and } \tau'\nu'\in\phi+W(0)\} \\ &= \tau L_Q((\tau')^{-1}\phi,(\tau')^{-1}W(0)). \end{aligned}$$

Since τ' acts on $\boldsymbol{C}'$ as a contractive linear transformation, we may assume that $\tau'W(\lambda)\subset W(0)$ and, hence, $W(\lambda)\subset(\tau')^{-1}W(0)$ for all $\lambda\in\Lambda$. Then we obtain

$$L_Q((\tau')^{-1}\phi,(\tau')^{-1}W(0))\supset L_Q((\tau')^{-1}\phi,\{W\})$$

and, consequently,

$$L_Q(\phi,\{W\})\supset\tau L_Q((\tau')^{-1}\phi,\{W\}).$$

This proves self-similarity of $L_Q(\phi,\{W\})$ with the ratio being equal to a PV number in *J*.

In this self-similarity, the sublattice $L_Q^{(0)}(\phi, W(0))$ of $L_Q(\phi,\{W\})$ has been chosen uniquely from the assumption that $W(0)$ is the largest window. However, it can be shown that there are q different ways of choosing from $L_Q((\tau)^{-1}\phi,(\tau')^{-1}W(0))$ a sublattice which is locally isomorphous to $L_Q(\phi,\{W\})$.

In the above discussion, we have tacitly assumed that the PV number related to self-similarity of $L_Q(\phi,\{W\})$ belongs to the same ideal $\boldsymbol{J}$ used in constructing $L_Q(\phi,\{W\})$. However, we can choose a PV number ρ belonging to a different ideal $\boldsymbol{J}'$. Self-similarity in this general case can be proved in a similar way. The case where $\boldsymbol{J}\cap\boldsymbol{J}'=\{0\}$ is of a particular interest because ρ induces, then, a permutation among different sublattices in (4*a*). We shall not, however, discuss this subject any further now but leave it to a study in the next section.

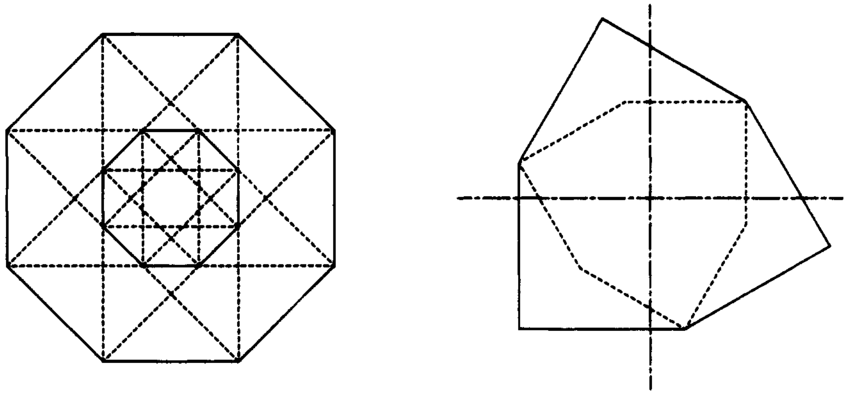
4. Examples

In this section, we shall apply the theory developed in § 3 to the cases of $n=8$, 10 and 12, in which cases $m=2$. We shall discuss each case separately. Several results in II will be used.

4.1. The case of an octagonal quasilattice ($n=8$)

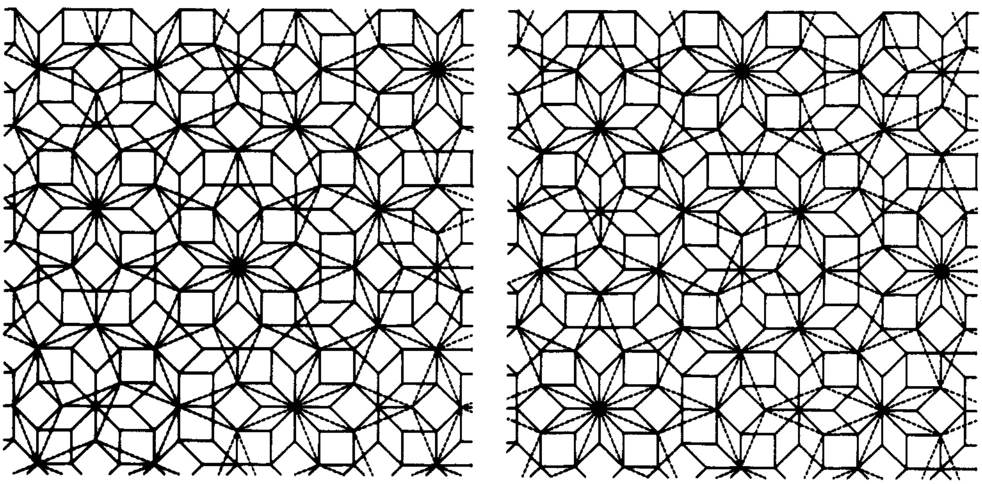
In this case, $\zeta=\exp(\pi\,\mathrm{i}/4)=(1+\mathrm{i})/\sqrt{2}$ and $\zeta'=-\zeta$. The fundamental PV unit is $\tau_0=1+\sqrt{2}$ $(=1+\zeta+\zeta^{-1})$, the silver ratio. The non-trivial ideal with the smallest norm is generated by $\tau=1+\zeta$ with $N(\tau)=2$. τ is a PV number; $|\tau|=(2+\sqrt{2})^{1/2}$ $(=1.8478)$, $|\tau'|=|1-\zeta|=(2-\sqrt{2})^{1/2}$ $(=0.7654)$ and $\arg\tau=\pi/8$. $K=\pi^{-1}(\boldsymbol{J})$ with $\boldsymbol{J}=\tau\boldsymbol{Z}(\zeta)$ is a face-centred hypercubic lattice in four dimensions.

The Voronoi polytope of the lattice point at the origin of L is projected by π' to a regular octagon whose vertices are at $\zeta^k\tau/\sqrt{2}$, $k=0,1,\ldots,7$ in $\boldsymbol{C}'$ as given in figure 1(*a*). If we take this polygon as the window W, the resulting octagonal quasilattice yields a tiling with square tiles and rhombic tiles. This quasilattice is inflated to two equivalent quasilattices with the complex self-similarity ratio $1+\zeta$ as shown in figure



(a) (b)

Figure 1. (a) The regular octagon represents the window used to construct an octagonal quasilattice. Subwindows partitioned by broken lines are associated with different types of the vertices of the quasilattice; the coordination number ranges from 3 to 8 and an outermost (or the central) subwindow is associated with 3-vertices (or with 8-vertices). The inner regular octagon (full lines) represents the narrowed window resulting in an inflated octagonal quasilattice with the ratio $1+\sqrt{2}$. It cuts right in two a subwindow associated with 5-vertices. It coincides with a subwindow associated with 8-coordinated vertices with respect to the 'second neighbour' coordination. (b) The full lines represent a window assigned to one of the four sublattices of a 'non-Bravais-type' dodecagonal quasilattice; other three windows are given by rotating it through multiples of $\pi/6$. The broken lines represent a window to be used on inflating the quasilattice with the ratio $1+\sqrt{3}$. The axes of the internal space are represented by chain lines.



(a) (b)

Figure 2. An octagonal quasilattice (full lines) and its two equivalent inflations (broken lines) with the complex self-similarity ratio $1+\zeta$. The directions of the bonds in the inflated quasilattices are rotated through $\pi/8 = \arg(1+\zeta)$ from those in the original quasilattice.

2. The two inflations have no common vertices because they are derived from the two different sublattices of L.

If the inflation is performed twice as given in figure 3, the resulting quasilattice is scaled by $2+\sqrt{2}$ from the original one because $(1+\zeta)^2=(2+\sqrt{2})\zeta$. Self-similarity of an octagonal quasilattice with this ratio was reported by Watanabe *et al* (1987). Their octagonal quasilattice is, however, different from the one in figure 3; theirs is obtained by the deflation method. Unfortunately, the rhombic tile cannot be uniquely deflated by their deflation procedure. Therefore, it is not warranted that the diffraction pattern of their quasilattice does not include a diffuse scattering.

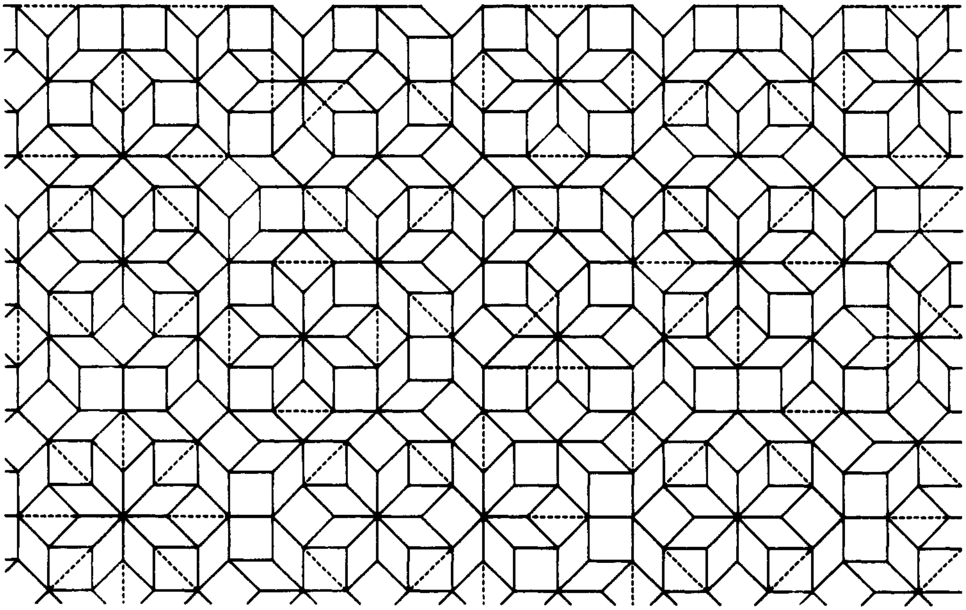


Figure 3. An octagonal quasilattice (full lines) and its inflation with the ratio $2+\sqrt{2}$ (broken lines). The decoration of a square tile in the inflated lattice (tiling) is not unique contrary to the octagonal quasilattice obtained by Watanabe *et al* (1987).

4.2. *The case of a decagonal quasilattice ($n=10$)*

In this case, $\zeta=\exp(\pi i/5)$ and $\zeta'=\zeta^3$. The fundamental PV unit is $\tau_0=(1+\sqrt{5})/2$ $(=\zeta+\zeta^{-1})$, the golden ratio. The non-trivial ideal with the smallest norm in $\boldsymbol{Z}(\zeta)$ is given by $\boldsymbol{J}=(\zeta+1)\boldsymbol{Z}(\zeta)$ with $N\boldsymbol{J}=5$ and $\Lambda\equiv\boldsymbol{Z}(\zeta)/\boldsymbol{J}=\boldsymbol{Z}_5=\{2,1,0,-1,-2\}$. The fundamental PV generator of J is given by $\tau=\tau_0(1+\zeta)$; $|\tau|=2\tau_0\cos(\pi/10)$ $(=3.078)$, $|\tau'|=2|\tau_0'|\sin(3\pi/10)$ $(=0.7265)$ and $\arg\tau=\pi/10$. Therefore, a decagonal quasilattice may be inflated to five equivalent sublattices with the complex ratio τ.

There exist several kinds of decagonal quasilattices but we consider here only the anti-Penrose lattice, which is a 'non-Bravais-type' quasilattice given by (4) with $W(0)$, $W(\pm1)$ and $W(\pm2)$ being equal to a decagon, truncated pentagons and small pentagons, respectively (Pavlovitch and Kléman (1987), see also II). We show in figure 4 the anti-Penrose lattice (tiling) and one of its five inflations with τ. Note that the lattice

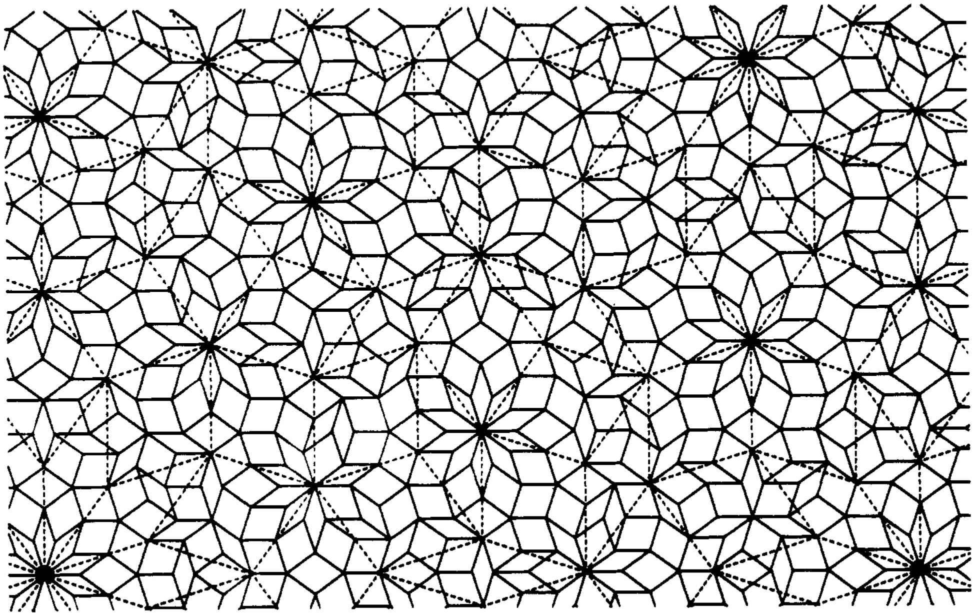


Figure 4. The anti-Penrose-type decagonal quasilattice (full lines) and its inflation with the ratio $(1+\zeta)(1+\sqrt{5})/2$ (broken lines).

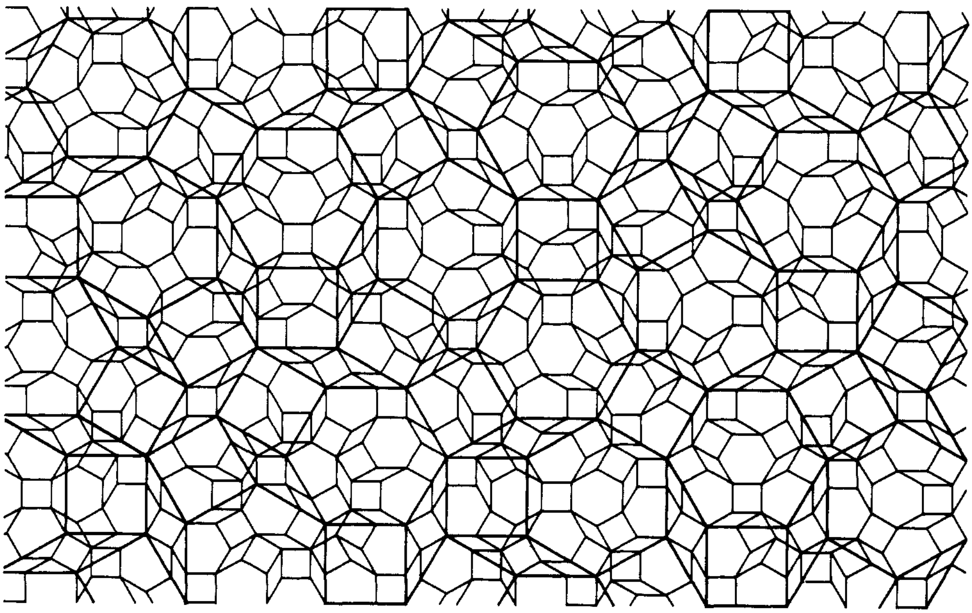


Figure 5. A dodecagonal quasilattice (thin lines) and its inflation with the ratio $1+\sqrt{3}$ (bold lines). The lattice points of the inflated quasilattice come from all the four sublattices into which the original quasilattice is divided.

points in the inflated quasilattice are derived only from the sublattice $L_Q^{(0)}(\phi, W(0))$ in the original quasilattice.

4.3. *The case of a dodecagonal quasilattice (n = 12)*

In this case, $\zeta = \exp(\pi i/6)$ $(=(\sqrt{3}+i)/2)$ and $\zeta' = -\zeta$. The fundamental PV unit is given by $\tau_0 = 1+\zeta = \tau_P \exp(\pi i/12)$ with $\tau_P = 2\cos(\pi/12) = (\sqrt{3}+1)/\sqrt{2}$ $(=1.932)$ being the platinum ratio. The non-trivial ideals with the smallest and the second-smallest norms in $\boldsymbol{Z}(\zeta)$ are $\boldsymbol{J}_1 = (1+\zeta^3)\boldsymbol{Z}(\zeta)$ and $\boldsymbol{J}_2 = (\zeta+\zeta^{-1})\boldsymbol{Z}(\zeta)$ $(=\sqrt{3}\,\boldsymbol{Z}(\zeta))$, respectively, where $N\boldsymbol{J}_1 = 4$ and $N\boldsymbol{J}_2 = 9$. The fundamental PV generators of $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are given by $\tau_1 = 1+\sqrt{3}$ $(=1+\zeta+\zeta^{-1})$ and $\tau_2 = \sqrt{3}\,\tau_0$, respectively. Thus, a dodecagonal quasilattice may be inflated to four (or nine) equivalent quasilattices with the ratio τ_1 (or τ_2). Note that $\boldsymbol{J}_1 \cap \boldsymbol{J}_2 = \{0\}$.

We present here only one example, which is shown in figure 5. The dodecagonal quasilattice in figure 5 is a 'non-Bravais-type' quasilattice (Niizeki 1988) obtained from a dodecagonal lattice (the hyperhexagonal lattice) L in four dimensions by assigning windows of trigonal hexagons (see figure 1(b)) with different orientations to four of the nine sublattices into which L is divided with $K_2 = \pi^{-1}(\boldsymbol{J}_2)$ (the other five sublattices are discarded). The inflation in figure 5 is performed with PV number τ_1. Therefore, the quasilattice is divided into four sublattices by $\boldsymbol{J}_1$ and only one of the four is retained in the first step of the inflation procedure. Then, each window is multiplied by τ_1' $(=1-\sqrt{3})$ and, consequently, is narrowed as presented in figure 1(b); this is because the four sublattices with non-empty windows are left invariant by the linear transformation $\tilde{\tau}_1 = \pi^{-1}(\bar{\tau_1}\cdot)$ (on account of $\psi(\tau_1) = 1+i+i^{-1} = 1$; for ψ see II).

5. Discussions

We shall argue the difference between the self-similarity investigated in this paper and that in I and II. We begin with investigating the feature of the self-similarity associated with a PV unit. On the inflation of the Penrose lattice with the ratio $(1+\sqrt{5})/2$, whether a lattice point is to be retained or discarded is determined uniquely by the type of the relevant vertex (de Bruijn 1981). The same is true for the case of the inflation of the dodecagonal quasilattice in figure 5 with the ratio $2+\sqrt{3}$ (Niizeki 1988). This is not the case for the inflation of the octagonal quasilattice with the PV unit $1+\sqrt{2}$ as given in figure 6 because a half of the five-coordinated vertices (5-vertices) are retained and the other half are discarded as explained in figure 1(a). Fortunately, the two groups of 5-vertices can be distinguished if we closely observe figure 6: (i) 5-vertices appear in pairs associated with bonds shared by pairs of square tiles, (ii) one of the 5-vertices in a pair is retained on the inflation but the other is discarded; (iii) a 5-vertex to be retained is located at the centre of a regular octagon formed by the 'second neighbours' but the one to be discarded is not. In fact, the narrowed window coincides with the subwindow corresponding to 8-vertices with respect to the 'second neighbour' coordinations (see figure 1(a)), so that observation (iii) applies as a rule of the inflation for all the vertices independently of their coordination numbers. Incidentally, we remark that the central subwindow is exactly equal to $(1+\sqrt{2})^{-2}$ times the original window, so that a double inflation of the octagonal quasilattice with $1+\sqrt{2}$ is obtained by retaining their 8-vertices only.

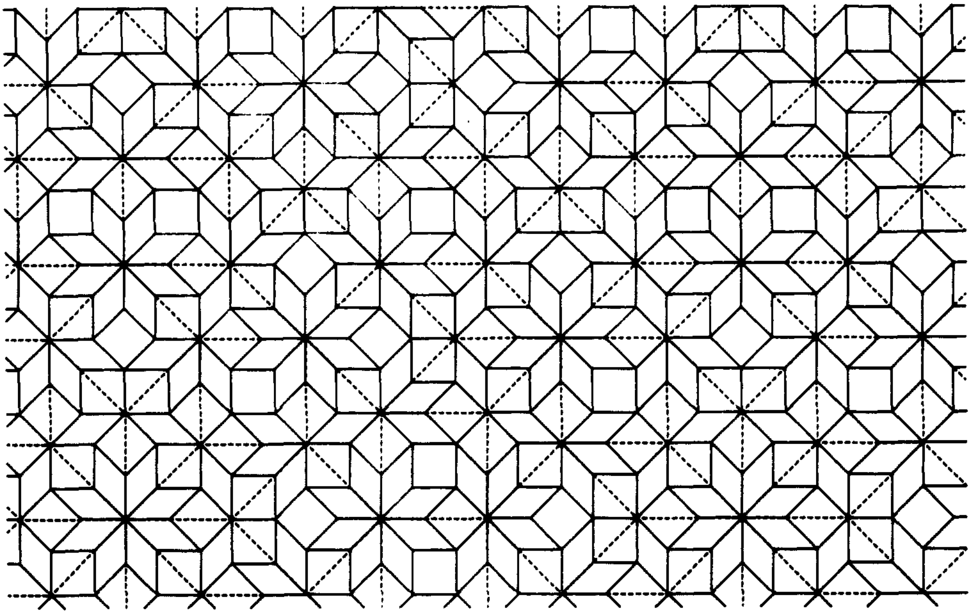


Figure 6. An octagonal quasilattice (full lines) and its inflation (broken lines) with a PV unit $1+\sqrt{2}$. The vertices to be retained on the inflation are determined by their local configurations.

It is a general feature of the inflation with a PV unit that a local configuration of a lattice point determines uniquely whether it is to be retained or not on the inflation. This is derived by that the inflation rule is formulated in the projection method solely by narrowing the window(s). On the contrary, whether a lattice point is to be retained or not on the inflation with a non-unit PV number is not determined only by its local configuration on account of the non-uniqueness of the inflation. Therefore, this self-similarity is of less physical importance than the one in the case of a PV unit is. This is supported also by the fact that self-similarity associated with a non-unit PV number is not clearly observed in the diffraction pattern, in contrast to the case of a PV unit. In view of the important difference between the two types of self-similarities, we shall distinguish between them by calling the self-similarity associated with a PV unit as the type I self-similarity and that with a non-unit PV number as type II.

Let $\tau \in \boldsymbol{Z}(\zeta)$. Then the external space and the internal one are invariant subspaces of $E_{2m} \simeq \boldsymbol{C}^m$ against the linear transformation $\tilde{\tau}$ represented by the diagonal matrix $(\tau, \tau', \ldots, \tau^{(m-1)})^{\text{diag}}$. The condition for τ to be a PV number is equivalent to $\tilde{\tau}$ acting on the external space as an expansive similarity transformation but on the internal one as a contractive linear transformation. Moreover, $\tilde{\tau}$ transforms the basis vectors of the n-gonal lattice L among themselves as $\tilde{\tau}(a_0, a_1, \ldots, a_{(2m-1)}) = (a_0, a_1, \ldots, a_{(2m-1)})M$ with M being a non-singular integer matrix. It follows that $\det M = N(\tau)$. If τ is a unit then $N(\tau) = 1$ and M is unimodular. Otherwise, $q = N(\tau)$ is a larger positive integer than 1. Thus, $\tilde{\tau}$ leaves L invariant if τ is a unit but changes it into $K = \pi^{-1}(\tau \boldsymbol{Z}(\zeta))$, i.e. one of the q equivalent sublattices of L. It can be concluded generally that a self-similarity of a quasilattice in any dimensions belongs to type I or II according to whether the relevant transformation acting on the starting higher-dimensional lattice is unimodular or not, respectively (for the case of type I self-similarity, see Katz and Duneau (1986) and Gähler (1986)).

Another difference between the two types of self-similarity is discussed by Pleasants (1984) on the Pleasants' patterns.

Every periodic lattice has an infinite variety of superlattices which are similar to itself. The inflation in a self-similarity in this sense is not unique because every superlattice has several equivalent companions. Thus, self-similarity of a periodic lattice is considered to belong to type II. This view is also consistent with type II self-similarity of a quasilattice, because it is derived from the presence of several equivalent sublattices of the starting lattice. On the contrary, type I self-similarity is peculiar to a quasilattice.

Katz and Duneau (1986) claimed that the inflation of the type I self-similarity is not unique in some cases. Their claim is based on observation that even if the narrowed window $\tau' W$ on the inflation is translated in the internal space by an arbitrary vector η under the condition that $\eta + \tau' W \subset W$, the resulting quasilattice $L_Q(\phi - \eta, \tau' W)$ as well as $L_Q(\phi, \tau' W)$ is similar to the original quasilattice $L_Q(\phi, W)$. In this case, however, the centre of the new window $\eta + \tau' W$ is shifted by η from that of the original window W, so that the relative configuration between the two windows is unsymmetrical. This causes a lattice point in a local configuration to be retained on the inflation, while another one in the same local configuration, except for its orientation, is discarded. Such an inflation will be of no physical interest. If we restrict our concern only to the symmetrical inflation assumed in this series of papers, the uniqueness of the inflation in type I self-similarity is restored.

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